

# Kolmogorov's law for two-dimensional electron-magnetohydrodynamic turbulence

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## Abstract

The analogue of the Kolmogorov's four-fifths law is derived for two-dimensional, homogeneous, isotropic EMHD turbulence in the energy cascade inertial range. Direct numerical simulations for the freely decaying case show that this relation holds true for different values of the adimensional electron inertial length scale,  $d_e$ . The energy spectrum is found to be close to the expected Kolmogorov spectrum.

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The statistical theory of three-dimensional fully developed hydrodynamic turbulence relies on one outstanding issue: the nonlinear transfer of energy from large to small scales [1–3]. It is therefore interesting to look for two-dimensional turbulent fluid dynamical systems sharing this same feature. Actually many of them exhibit a reversed energy flux, from the small scales to the larger ones, as is the case of 2D Navier-Stokes turbulence [4–7], Hasegawa-Mima turbulence [8] or its geophysical counterpart, equivalent barotropic turbulence [9].

In this framework 2D electron-magnetohydrodynamic (EMHD) turbulence deserves special attention, beyond its modeling applications, since it has been shown to display, for the freely decaying case, a forward energy cascade á la Richardson-Kolmogorov [10].

In this Letter it is introduced a relation which is the counterpart of the Kolmogorov four-fifths law for homogeneous and isotropic 2D EMHD turbulence. Its content is compared with the results obtained by direct numerical simulations.

EMHD equations are a fluid dynamical model for a cold electron plasma, moving in a uniform charge-neutralizing background of stationary ions. In recent years this model has received considerable interest for its relation to inertially confined plasma and to laser-plasma interactions, but the comparison with experimental results is limited by the fact that plasma which evolve according to EMHD equations are usually short-lived.

The equations for the electron plasma in adimensional form are [11]

$$d_e^2 \frac{\partial \mathbf{v}}{\partial t} + d_e^2 \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p - \mathbf{E} - \mathbf{v} \times \mathbf{B} + \nu_q (-\nabla)^{2q} \mathbf{v} . \quad (1)$$

The velocity field  $\mathbf{v}$ , the magnetic field  $\mathbf{B}$ , and the electric field  $\mathbf{E}$  are allowed to have a nonzero component in any direction of three-dimensional space, meanwhile their functional dependence is restricted to the plane coordinates ( $\frac{\partial}{\partial x_3} \equiv 0$ ). The equations have been adimensionalized with respect to the characteristic macroscopic length  $L$ , the typical magnetic field  $B_0$ , the characteristic time  $\tau = m_e c / (e B_0 d_e^2)$  and  $d_e = [m_e c^2 / (4\pi e^2 n L^2)]^{1/2}$  is the ratio of the inertial electron length scale to the integral scale  $L$ . The density of the number of electrons  $n$  is assumed to be uniform according to the incompressibility of the velocity field  $\nabla \cdot \mathbf{v} = 0$ . The approximation made by considering motionless ions requests that the adimen-

sional ion inertial length,  $d_i$ , must be larger than unity, thus limiting the range of admissible values for  $d_e = d_i(m_e/m_i)^{1/2}$  to the interval  $d_e \gtrsim 0.02$ . As long as we are dealing with a fluid description of plasma, all the lengthscales under consideration must largely exceed the Larmor radius. For stationary ions and negligible displacement current, the Ampère law becomes

$$\mathbf{v} = -\nabla \times \mathbf{B} . \quad (2)$$

The last term in equation (1) is a dissipative term which mimics the effects of electron viscosity ( $q = 1$ ) or resistivity ( $q = 0$ ). The total energy of the electron fluid

$$E = \frac{1}{2} \int (d_e^2 \mathbf{v}^2 + \mathbf{B}^2) d^2x \quad (3)$$

is conserved by these equations in the ideal, non-collisional case.

In the spirit of Kolmogorov analysis [1,2], one takes under consideration the spectral energy budget

$$\frac{\partial}{\partial t} \mathcal{E}(K) + \Pi(K) = -\mathcal{D}(K) \quad (4)$$

where  $\mathcal{E}(K) = \int_0^K E(k) dk$  is the mean cumulative energy per unit mass contained at wavenumbers smaller than  $K$ , and  $E(k)$  is the energy spectrum.  $\Pi(K)$  is the energy flux (per unit mass) from wavenumbers  $k \leq K$  to larger wavenumbers, and  $\mathcal{D}(K)$  is the cumulative energy dissipation up to wavenumber  $K$ . Since the dissipation is localized to high wavenumbers  $k \geq K_d$ , there is a range of wavenumbers  $K_0 \ll K \ll K_d$  where  $\mathcal{D}(K) \simeq 0$ , and the energy flux  $\Pi(K)$  is determined by the inertial transfer of energy from the energy containing eddies at wavenumbers around  $K_0$ . When the large scale energy input due to the straining of energy containing eddies at scale  $K_0$  is equilibrated by dissipation taking place at small scales, one expects the energy flux through wavenumber  $K$ ,  $\Pi(K)$ , to be independent of  $K$  [3]. Actually it must be remarked that, due to the energy decay, the flux approaches a constant value, equal to the total dissipation, only for very large  $K$ , and the crossover to the asymptotic behavior is very slow [12]. In an analogous fashion as in

three-dimensional hydrodynamic turbulence, in the limit of vanishing viscosity,  $\nu \rightarrow 0$ , the energy flux is expected to achieve a finite positive limit, depending on the value of  $d_e$ ,  $\bar{\varepsilon} > 0$ ,

$$\Pi(K) \simeq \bar{\varepsilon} \quad K_0 \ll K \ll K_d. \quad (5)$$

This is a strong request which in 3D-hydrodynamics has experimental evidence; in the case under consideration it will be shown that numerical simulations provide reasonable support to this hypothesis.

Assuming statistical homogeneity the energy flux,  $\Pi(K)$  can be expressed by means of physical space statistics by performing the Fourier transform of (4). The result is

$$\Pi(K) = \frac{1}{2\pi} \int d^2\ell \, K \frac{J_1(K\ell)}{\ell} \varepsilon(\ell). \quad (6)$$

where  $J_1$  is the first-order Bessel function of the first kind, and the energy flux in the physical space is given by

$$\varepsilon(\ell) = -\frac{\partial}{\partial t} \frac{1}{2} \langle d_e^2 \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x} + \ell) + \mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x} + \ell) \rangle|_{NL} \quad (7)$$

where the subscript  $NL$  stands for the nonlinear contribution to the time derivative of the fields, as it can be extracted by the equations of motion (1), and the brackets  $\langle \dots \rangle$  express ensemble averages. Using (1) and (2), making repeatedly use of statistical homogeneity, of incompressibility of the velocity field and solenoidality of the magnetic field, one obtains the following relation

$$\begin{aligned} \varepsilon(\ell) = & \frac{\partial}{\partial \ell_i} \left( -\frac{d_e^2}{4} \langle (\delta \mathbf{v} \cdot \delta \mathbf{v}) \delta v_i \rangle + \right. \\ & \left. \frac{1}{4} \langle (\delta \mathbf{v} \cdot \delta \mathbf{B}) \delta B_i \rangle - \frac{1}{8} \langle (\delta \mathbf{B} \cdot \delta \mathbf{B}) \delta v_i \rangle \right) \end{aligned} \quad (8)$$

with  $i = 1, 2$  denotes the planar components. The expression (8) for the physical space energy flux is the analogue of the Kármán-Howarth-Monin relation [2], and it involves only differences of dynamical fields as  $\delta \mathbf{v} = \mathbf{v}(\mathbf{x} + \ell) - \mathbf{v}(\mathbf{x})$ . To proceed further one assumes statistical isotropy and it is then possible to show that the physical space energy flux is

$$\begin{aligned} \varepsilon(\ell) = & \frac{1}{4} (2 + \ell \partial_\ell) \left\{ -\frac{d_e^2}{3} (4 + \ell \partial_\ell) \frac{S_3(\ell)}{\ell} - d_e^2 \frac{V_3(\ell)}{\ell} + \right. \\ & \left. \frac{1}{2} (2 + \ell \partial_\ell) \frac{T_3(\ell)}{\ell} - \frac{U_3(\ell)}{\ell} + \frac{W_3(\ell)}{\ell} - \frac{1}{2} \frac{X_3(\ell)}{\ell} \right\} \end{aligned} \quad (9)$$

in which appear the following third order structure functions

$$\begin{aligned}
S_3(\ell) &= \langle \delta v_{\parallel} \delta v_{\parallel} \delta v_{\parallel} \rangle, & T_3(\ell) &= \langle \delta v_{\parallel} \delta B_{\parallel} \delta B_{\parallel} \rangle, \\
U_3(\ell) &= \langle \delta v_{\parallel} \delta B_{\perp} \delta B_{\perp} \rangle, & V_3(\ell) &= \langle \delta v_3 \delta v_3 \delta v_{\parallel} \rangle, \\
W_3(\ell) &= \langle \delta v_3 \delta B_3 \delta B_{\parallel} \rangle, & X_3(\ell) &= \langle \delta B_3 \delta B_3 \delta v_{\parallel} \rangle,
\end{aligned} \tag{10}$$

where the standard notation for longitudinal,  $\delta v_{\parallel} = \delta v_i \ell_i / \ell$ , and transverse differences,  $\delta v_{\perp} = \epsilon_{ij} \delta v_i \ell_j / \ell$ , has been used.

As a consequence of hypothesis (5), it can be shown by a saddle-point argument that the physical space flux,  $\varepsilon(\ell)$ , must behave as

$$\varepsilon(\ell) \simeq \bar{\varepsilon} \quad \lambda \ll \ell \ll \ell_0, \tag{11}$$

in the limit of vanishing viscosity, where the inertial range of lengthscales is now delimited by the ‘‘Taylor scale’’,  $\lambda = (\nu_q E / D)^{1/2q}$ , where  $D$  is the energy dissipation, and the energy containing scale  $\ell_0 \sim 1/K_0$ . Inserting the expression for the energy flux (9) inside relation (11), one obtains the 2D EMHD counterpart of Kolmogorov’s four-fifths law [1,2]

$$Q_3(\ell) \simeq \bar{\varepsilon} \ell \tag{12}$$

where

$$\begin{aligned}
Q_3(\ell) &= -\frac{2}{3} d_e^2 S_3(\ell) - \frac{1}{2} d_e^2 V_3(\ell) + \frac{1}{2} T_3(\ell) \\
&\quad - \frac{1}{2} U_3(\ell) + \frac{1}{2} W_3(\ell) - \frac{1}{4} X_3(\ell).
\end{aligned} \tag{13}$$

Relation (12) relies only on the aforementioned hypothesis here recalled: homogeneity, isotropy, and the existence of an inertial range of wavenumbers in which the energy flux is constant, with a value tending to a finite positive limit for vanishing viscosity. The most remarkable aspect of relation (12,13) is that it does not only provide a linear scaling for the third order structure function  $Q_3(\ell)$  within the inertial range of lengthscales, but it also prescribes the value of the numerical coefficient appearing in front of the scaling relation. Moreover, it is valid for any value of  $d_e$ , meanwhile no power law scaling relation is expected

to hold for, say, the second order structure functions, apart from limiting cases such as  $d_e \sim 1$  and  $d_e \ll 1$  [10].

To check the validity of EMHD Kolmogorov law (12), the equations (1) are solved in a square box of size  $2\pi \times 2\pi$  imposing periodic boundary conditions, by means of a standard pseudospectral method with resolution  $N \times N$ . Hyperdissipation with  $q = 4$  is employed in order to achieve a larger extent of the inertial range, which in physical space is known to be much narrower than in spectral space [2]. Hyperviscosity is set to  $\nu_4 = 10^{-12}, 10^{-13}$  for  $N = 512$  simulation and  $\nu_4 = 10^{-13}$  for  $N = 1024$ .

The initial conditions are  $v(k) \sim k^2 \exp(-k^2/2k_0^2)$  with random phases,  $k_0 = 1$  and total energy of order unity in both resolution simulations. After a transient of a few large eddy turnover times, when the energy initially contained at the lowest wavenumbers starts to cascade down to small scales, the energy dissipation reaches a maximum value and then a self similar stage of decay sets in [10]. The energy flux is approximately constant throughout the inertial range of wavenumbers (see Fig.1), and, in agreement with the assumption (5), its value appears to be asymptotically independent of viscosity. The structure function  $Q_3(\ell)$  is computed during the self similar stage of decay. The results are obtained after averaging over a short time in order to get better statistics at small scales. As shown in Fig.2 the compensated structure function  $Q_3(\ell)/(\bar{\epsilon}\ell)$  approaches unity, as prescribed by the relation (12) in an interval delimited from below by the ‘‘Taylor scale’’  $\lambda$  and above by the energy containing scale  $\ell_0$ . By lowering the viscosity (crosses,  $N = 1024$ ) the scaling range extends to smaller scales over almost one decade. As previously remarked, the width of the inertial range is actually diminished by the fact that, for a decaying flow, the energy flux in wavenumber space is not constant except asymptotically (see Fig.1). This results are an evident numerical confirmation of the validity of the Kolmogorov-type relation (12). This kind of assessment is important since it lies at the foundations of the statistical study of turbulence.

Introducing the further hypothesis of statistical self-similarity one can infer the following scaling behavior for the velocity and for the magnetic field differences in the asymptotic case

$$\begin{aligned}\delta v(\ell) &\propto \bar{\varepsilon}^{1/3} \ell^{1/3} \\ \delta B(\ell) &\propto \bar{\varepsilon}^{1/3} \ell^{4/3}\end{aligned}\quad \ell \ll d_e , \quad (14)$$

which leads to the small scales energy spectrum, dominated by kinetic energy,

$$E(k) = C_K \bar{\varepsilon}^{2/3} k^{-5/3} \quad kd_e \gg 1 . \quad (15)$$

On the other hand, for scales larger than  $d_e$ , the expected self-similar scaling is

$$\begin{aligned}\delta v(\ell) &\propto \bar{\varepsilon}^{1/3} \ell^{-1/3} \\ \delta B(\ell) &\propto \bar{\varepsilon}^{1/3} \ell^{2/3}\end{aligned}\quad \ell \gg d_e \quad (16)$$

leading to the energy spectrum, dominated by magnetic energy,

$$E(k) = C'_K \bar{\varepsilon}^{2/3} k^{-7/3} \quad kd_e \ll 1 . \quad (17)$$

The slopes of the computed spectra, as shown in Fig.3, are close to the estimates (15) and (17) over a wide range of wavenumbers.

The main results of this work are the derivation of a Kolmogorov-type relation for 2D EMHD decaying turbulence, and its numerical confirmation. Since the Kolmogorov's law stands as a corner stone in the study of the statistical features of turbulence, these results form the basis of further analysis, starting from the issue of intermittency.

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## REFERENCES

- [1] A.N. Kolmogorov, Dokl. Akad. Nauk SSSR **32**, 16 (1941)
- [2] U. Frisch, *Turbulence. The legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995)
- [3] G. K. Batchelor, *The theory of homogeneous turbulence* (Cambridge University Press, Cambridge, 1953)
- [4] J. C. McWilliams, J. Fluid Mech. **146**, 21 (1984).
- [5] M. Brachet, M. Meneguzzi, H. Politano, and P. Sulem, J. Fluid Mech. **194**, 33 (1988)
- [6] B. Legras, P. Santangelo, and R. Benzi, Europhys. Lett. **5**, 37 (1988)
- [7] L. M. Smith and V. Yakhot, Phys. Rev. Lett. **71**, 352 (1993)
- [8] N. Kukharkin, S. A. Orszag, and V. Yakhot, Phys. Rev. Lett. **75**, 2486 (1995)
- [9] V. D. Larichev and J. C. McWilliams, Phys. Fluids A **3**, 938 (1991)
- [10] D. Biskamp, E. Schwarz, and J. F. Drake, Phys. Rev. Lett **76**, 1264 (1996).
- [11] A. S. Kingsep, K. V. Chukbar, and V. V. Yan'kov, in *Reviews of Plasma Physics* (Consultants Bureau, New York, 1990), Vol.16, 243.
- [12] V. Borue and S. A. Orszag, Phys. Rev. E **51**, 857 (1995)



## FIGURES

FIG. 1. Energy flux in wavenumber space normalized to energy dissipation. (a)  $d_e = 0.3$ . Continuous line:  $N = 512$ ;  $\nu_4 = 10^{-12}$ . Dotted line:  $N = 1024$ ;  $\nu_4 = 10^{-13}$ . (b)  $d_e = 0.02$ . Continuous line:  $N = 512$ ;  $\nu_4 = 10^{-13}$ .

FIG. 2. Compensated structure function  $Q_3(\ell)/(\bar{\varepsilon}\ell)$ . (a)  $d_e = 0.3$ ; Diamonds:  $N = 512$ ;  $\nu_4 = 10^{-12}$ . Crosses:  $N = 1024$ ;  $\nu_4 = 10^{-13}$ . (b)  $d_e = 0.02$ ; Diamonds:  $N = 512$ ;  $\nu_4 = 10^{-13}$ .

FIG. 3. Energy spectrum. (a)  $d_e = 0.3$ ; Diamonds:  $N = 512$ ;  $\nu_4 = 10^{-12}$ . Crosses:  $N = 1024$ ;  $\nu_4 = 10^{-13}$ . The continuous line is the Kolmogorov spectrum (15) with  $C_K = 2.0$ . (b)  $d_e = 0.02$ ; Diamonds:  $N = 512$ ;  $\nu_4 = 10^{-13}$ . The continuous line is the spectrum (17) with  $C'_K = 8.0$





